TITLE: Theory of Human Vibration Response

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ABSTRACT:

Analytical and experimental studies of whole body human dynamics under random vibration are presented.  A previously unreported criteria "absorbed power" is developed through the application of transfer functions.  This method is applicable to assess the effects of stationary and non-stationary vibration records.  It is equally effective for synthesis or analysis.  The linearity of human response to vibration is established on a qualitative basis.

Examination of the "absorbed power" criteria indicates advantages not present in the acceleration measurement.  "Absorbed power" is a scalar quantity which may be described by magnitude only.  It is additive and may be summed in multi-degree-of-freedom environments.

"Absorbed power" does not require frequency spectrum analysis.  Optimization studies involving human dynamics may be conducted continuously without the time lapse that occurs for frequency spectrum analysis.  The findings of this research indicate with substantial credence that "absorbed power" and transfer function techniques may provide the basis for a universally usable human vibration measurement method which shall be applicable to air, sea, and land transportation media.

# THEORY OF HUMAN VIBRATION RESPONSE

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Introduction: The subject of human vibration has been treated by many investigators with some of the earliest work dating back to 1818. Current bibliographies indicate over 1500 separate studies have been directed to the problem of identifying human response to vibration.

In 1962, it appeared that existing research would continue to place heavy emphasis upon experimental tests to assess the effects of mechanical vibration. New programs based upon analytical approaches were not appearing. Therefore, in January 1963, due to an extreme internal need for forecasting human response to vehicle vibration, the U.S. Army Tank-Automotive Center began a basic research program.

The premise of this work was oriented towards random vibration, as it has been generally recognized that random vibration best describes the environment created by military, air, sea, and automotive transportation.

This paper discusses whole body human response to mechanical vibration below 60 cps. The theoretical considerations are based on the view that man's response in a vibratory environment can be determined through measurement of input conditions only. Separation of the total problem (anatomical, psychological, and physiological) into this narrow premise has produced a new parameter identified as "absorbed power".

This parameter is completely different. It has been developed to describe human dynamics without prior knowledge of the frequency spectrum and can identify human response with single numerical distinctness.

$$\text{Average "absorbed power"} = \lim_{T \to \infty} \frac{1}{T} \int_0^T F(t)V(t)dt \quad (1)$$

Discussion: The word "random" is used primarily in a statistical sense and implies vibration which does not show frequency or amplitude order except as a statistic. Random vibration in general describes environmental excitations which

exist over a band of frequencies. When motion is periodic, the
accelerations are determinable by historical examination. When
the motion is random, non-periodic, only the probability of future
occurrence can be stated.

Previous to this research, to conveniently cope with
random vibration calculations and experimental work, the concept
known as acceleration density was used. In brief review, this
concept takes the square of the RMS value of acceleration over
specific frequency intervals and yields the mean square value of
the acceleration for that particular bandwith. If this mean squared
value of acceleration is divided by the bandwidth and the bandwith
is made to approach zero, the resulting value can be plotted versus
frequency over a spectrum of interest. This is known as the
POWER SPECTRAL DENSITY (PSD). The mechanical analogy of power is the
rate of doing work. PSD is proportional to the square of the
vibration amplitude. Hence, a plot of PSD versus frequency shows
the power distribution of the vibration environment. PSD and "ABSORBED
POWER" will again be frequently referred to. These two quantities
are not interchangeable. Meaning and application of these two terms
are distinctly different. PSD statistically describes the vibration.
"Absorbed Power" measures the rate at which the vibratory input is
used. Normally, comprehensive descriptions of random vibration tests
require PSD specifications. For "white noise" random vibration, the
PSD is flat; that is, all frequency components are present at equal
energy levels.

Power spectral density and other commonly used techniques such as time histories, peak distribution, or probability records describe the vibratory input to the "occupant" or of the "man" but fail to identify his subjective response to this environment.

Man has the capacity of evaluating situations involving touch, taste, sight, and sound. Other research has successfully established the property of "quantity" in many of these common sensations, but the problem of establishing a unit to quantify and measure the perception of vibration analytically has not been previously disclosed.

Approach: A principal concept in modern analysis of shock and vibration is the transfer function. This function is the complex ratio of output information to the input which caused the output.

The transfer function is usually denoted G(S) where $S = \sigma + j\omega$ in the Laplace transform of a function g(t). Sigma ($\sigma$) represents the transient term and $j\omega$, the steady state condition. The transform is denoted by the symbol L[g(t)] or G(S) and is defined by the integral,

$$L[g(t)] = G(S) = \int_0^\infty e^{-st} g(t)dt. \qquad (2)$$

General form of the transfer function G(S) is the ratio,

$$G(S) = \frac{\text{Response Function}}{\text{Input Function}} (S) \qquad (3)$$

Each combination of output and input in a linear system may be described by a transfer function. The transfer function relating the acceleration input $A_{in}$ to the output $A_{out}$ is given by Equation 4. The transfer function relating the force of vibration $F_{in}(S)$ to the output acceleration is given by Equation 5.



$$G_A(S) = \frac{A_{out}}{A_{in}}(S). \tag{4}$$

$$G_F(S) = \frac{F_{in}}{A_{out}}(S). \tag{5}$$

The first hypothesis of this program considered the application of transfer function techniques to human dynamics.

The essential constraint controlling the use of the transfer function is that the system in question responds linearily within the bounds of interest. Therefore, application of the transfer function technique to human dynamics required the establishment or acceptance that man behaves as a linear system.

The frequency spectrum is used to produce the transfer function. In this research program, a common procedure was used;

predetermined sinusoidal inputs of specified magnitude and frequency
were produced and the magnitude and phase of the steady-state output
recorded.

The necessary data for the transfer function was
collected in a series of experiments for sinusoidal and random
environments involving thirty-one male test subjects representing
enlisted military personnel from the U.S. 8th Artillery Group,
Detroit, Michigan, and civilian personnel from the U.S. Army Tank-
Automotive Center. The statistical summary of the test population
is provided in Appendix A.

The data collected described the input to the man
in terms of force, acceleration, and frequency. The output was
measured and recorded at the subject's head.

Figure 1 describes the vertical mode data as the two-
sided interval which can be expected to bracket the true mean with
90% confidence. The ordinate describes the RMS (root mean square)
value of acceleration in g's for 1 through 30 CPS under the following
test controls. The frequency was pre-set and remained fixed during
each run. The sinusoidal acceleration amplitude was increased
at a slow rate from a static position to vibration tolerance.
The tolerance criteria was based upon individual assessment
concerning a combination of: vibration severity, pain, loss of
physical control, or blurred vision. When tolerance was reached,
the subject actuated a buzzer which began the RMS data collection

procedure. Each subject was held at the tolerance level for 20 to 60 seconds. The time interval was dependent upon the stability of RMS data.

To gather response data concerning random vibration, experiments were conducted using uniform spectrum, white noise vibration filtered through 2 CPS and 10 CPS bandwidth filters. Random vertical and angular motion data were recorded and are reported in Figures 2 through 4.

The tolerance tests established the upper boundary of human response in accordance with the experimental control described. To validate the second hypothesis that whole body response is linear below this level, tests were performed at various acceleration levels down to very low intensity. Regression analysis indicated strong linearity throughout the frequency spectrum.

Later experiments approached linearity using random waveforms in a manner analogous to that of a constant rate spring. These experiments and others were sufficiently rigorous that within the reasonable constraints of our test, human response was sufficiently linear to determine the transfer function. The data used were mean values of pooled information collected at the seven different acceleration levels (0.35 to 2.0 g's P-P).

Figure 5 describes the transmissibility plot for mean values and the 90% confidence interval of head acceleration to the input acceleration.

The development of the analytical transfer function was achieved using Figure 5 and an asymptotic approximation method of the frequency spectrum.[2]

Equation 7 is the transfer function for the acceleration ratio described in Equation 6 where $C_1$ through $C_{12}$ are the constant coefficients listed in Table 1.

$$G_A(s) = \prod_{i=1}^{6} \quad G_i(s) = \frac{A_{output}}{A_{input}}(s) \tag{6}$$

$$G_A(s) = \frac{C_1 s^5 + C_2 s^4 + C_3 s^3 + C_4 s^2 + C_5 s + C_6}{s^6 + C_7 s^5 + C_8 s^4 + C_9 s^3 + C_{10} s^2 + C_{11} s + C_{12}} \tag{7}$$

TABLE 1

| CONSTANT | VALUE | CONSTANT | VALUE |
|---|---|---|---|
| $C_1$ | $11.28 \times 10^1$ | $C_7$ | $95.24 \times 10^1$ |
| $C_2$ | $10.40 \times 10^4$ | $C_8$ | $46.20 \times 10^4$ |
| $C_3$ | $38.70 \times 10^6$ | $C_9$ | $45.12 \times 10^6$ |
| $C_4$ | $16.08 \times 10^8$ | $C_{10}$ | $10.17 \times 10^9$ |
| $C_5$ | $44.31 \times 10^{11}$ | $C_{11}$ | $64.20 \times 10^{11}$ |
| $C_6$ | $98.41 \times 10^{11}$ | $C_{12}$ | $98.42 \times 10^{11}$ |

Satisfactory correlation of Equation 7 comparing analytical results with experimental measurements was achieved for random vibration, sinusoidal motion, squared shape inputs, and triangular waves. Correlation was secured in the time domain and for statistical samples (10 men).

Frequently, scaling of sensation or subjective response is approached by the method of fractionization. The stimulus is increased or decreased, and the subject reports the point where the sensation appears to halve, double, etc. In vibration environments, halving or doubling the stimulus does not halve or double the sensation. In human vibration, the factors which regulate sensation and which can be treated analytically have not been identified.

It was our third hypothesis that under vibration the human body may behave in an elastic or inelastic fashion. When the vibratory energy distorts the elastic body, dimensional changes take place producing reactions which tend to restore the body to the original position. The work that is performed in this process balances the applied load. Consequently, the body's elasticity produces restoring forces which are related to displacement. The body's vibratory motion continues until the energy imparted is dissipated or removed. The time rate at which this energy is used shall be referred to as "absorbed power". "Absorbed power" is a parameter relating vibratory input conditions to subjective response and is defined in Equation 1.

"Absorbed power" may be easily produced experimentally as follows: A time history signal of acceleration from magnetic tape or directly from an accelerometer (during field measurements) is sent into an electric circuit representing the product of the

$G_A(S)$ and the $G_F(S)$ transfer functions. The output from this circuit is force. The input acceleration record is also sent through an integrator producing velocity. The force and velocity signals are then multiplied electronically and put through an averaging circuit producing average "absorbed power". In preliminary experiments, it was verified that "absorbed power" was zero for artificial, inelastic systems, and changed exponentially for characteristics representing human response.

It was the objective in the initial tolerance tests (Figure 1) to secure subjective judgement based upon a uniform criteria. From these tests, mean values of acceleration at corresponding frequencies were used to develop a constant power curve of 5.3 watts. The electrical unit of "watts" was selected for initial use in that it was readily measured and described in comparison with other units of measure (HP and BTU). This curve was then fitted to the sinusoidal tolerance data. Results indicated that "absorbed power" displayed the same general characteristics as the judgement criteria.

The concept of "absorbed power" was then tested for sensitivity and variability. Experiments were designed to evaluate the ability to discriminate different vibration inputs and to order their comfort or severity.

The test that is reported here was performed using twenty-six random waveforms called "rides". Each ride

contained different PSD vibration characteristics in the frequency spectrum of 4 to 36 CPS at different levels of acceleration. The average acceleration used to order the rides was initially computed using four control subjects. The rides were presented in sequence of increasing acceleration magnitude. In essence, the first ride was empirically believed to be the smoothest and most comfortable and the last ride, the roughest. A group of ten subjects judged each ride (one minute duration) subjectively for comfort and reported only those rides that appeared to be "out of place". Acceleration (g's) and power (watts) were measured during the experiment.

The ten subjects reported disagreement in 60 instances with the acceleration ride sequence. By setting three or more reports per ride as an indication of significance, nine rides were repeatedly identified as misplaced. Two viewpoints developed: (1) either the rides were ordered incorrectly, or (2) the subjects were in error. The "absorbed power" data which was experimentally measured was then processed and compared to acceleration (Table 2). "Absorbed power" closely agreed with subjective response indicating the acceleration sequence was incorrect. In all but one case (Ride No. 10), the "absorbed power" measurement appeared to identify and discriminate ride comfort for small as well as large differences. It was observed that the measured input acceleration did not increase in precisely the same ascending order. This small

deviation was attributed to the increase in the test population from the original four control subjects to the final ten subjects.

TABLE 2
Acceleration - "Absorbed Power" Comparison

| Ride No. | Input (R.M.S.g's) | Measured Acceleration | No. of Calls | Power (Watts) |
|---|---|---|---|---|
| 1 | 0.417 | 0.430 | 0 | 0.1964 |
| 2 | 0.463 | 0.475 | 0 | 0.6864 |
| 3 | 0.479 | 0.480 | 5 | 0.1742 |
| 4 | 0.506 | 0.512 | 0 | 0.3183 |
| 5 | 0.555 | 0.559 | 4 | 0.1635 |
| 6 | 0.658 | 0.675 | 0 | 2.0976 |
| 7 | 0.660 | 0.672 | 6 | 0.3098 |
| 8 | 0.685 | 0.703 | 0 | 0.5649 |
| 9 | 0.812 | 0.828 | 7 | 0.2657 |
| 10 | 0.840 | 0.850 | 1 | 0.8080 |
| 11 | 0.841 | 0.881 | 0 | 3.6416 |
| 12 | 0.853 | 0.855 | 3 | 2.2843 |
| 13 | 0.929 | 0.924 | 5 | 0.6626 |
| 14 | 0.956 | 0.962 | 0 | 1.1632 |
| 15 | 1.007 | 1.025 | 2 | 0.6667 |
| 16 | 1.074 | 1.121 | 5 | 0.6203 |
| 17 | 1.144 | 1.164 | 0 | 5.6228 |
| 18 | 1.162 | 1.201 | 5 | 1.6706 |
| 19 | 1.194 | 1.205 | 0 | 1.7719 |
| 20 | 1.205 | 1.211 | 4 | 0.5415 |
| 21 | 1.208 | 1.210 | 0 | 0.9220 |
| 22 | 1.438 | 1.504 | 0 | 1.2092 |
| 23 | 1.475 | 1.628 | 10 | 0.8324 |
| 24 | 1.617 | 1.599 | 0 | 1.7269 |
| 25 | 1.808 | 1.854 | 0 | 4.0612 |
| 26 | 1.974 | 1.942 | 3 | 1.3736 |

To increase the utility of the "absorbed power" statement, Equation 1, research work was directed towards developing a simplified relationship which would not require stringent control of the input data and which also would include the effects of "time".

It was rationalized that either RMS of acceleration or power spectral density would handle all the input requirements

which may be defined in the time domain or statistically. Equation 8 is the new expression for "absorbed power". Detailed development is given in Appendix B. It describes the important relationship that the total "absorbed power" for a vibration spectrum is the summation of the power at each frequency. The power is computed as the product of the acceleration squared ($A_i^2$ r.m.s.) at each frequency by the new parameter ($K_i$). This parameter is a function of frequency but does not vary at any one frequency.

$$P_{(ave)} = \sum_{i=1}^{N} K_i A_i^2 \text{ r.m.s.} \tag{8}$$

Equation 8 describes power for short duration exposures where "absorbed power" is not affected by fatigue or time differences. Time alters "absorbed power" and this effect is included in Equation 9,

$$P_t = P + \frac{1}{t_o} \int P dt \tag{9}$$

$P_t$ is equal to the sum of the short term and time dependent "absorbed power". The time factor $1/t_o$ represents the approximate onset of fatigue and t is the time of the exposure. Power need not remain constant. It may be a function of time. However, where power is constant, the acceleration relationship is given as:

$$A_{r.m.s.} = \sqrt{\frac{\frac{P_t}{k}}{1 + t/t_o}} \tag{10}$$

Summary and Conclusions: The basic premise of this
research program has been to increase the knowledge and understanding
of human response to mechanical vibration. Initial research steps
were basic and conventional, and centered upon vertical sinusoidal
tolerance studies.

Experiments followed to obtain tolerance data for
random vibration in the vertical and the angular modes of pitch and
roll. After this accomplishment, a temporary impasse was reached.
The problem arose of describing random vibration environments which
may exist in an infinite number of patterns. PSD (power spectral
density) techniques appeared to be of minimal help. A new approach
was selected. The validity and applicability of transfer function
techniques to human dynamics was tested.

Success in relating analytical transfer function
statements and experimental test data was reported in May 1965.
Correlation was shown for white noise (random vibration) using
2 CPS and 10 CPS bandwidth filters.

The value of this analytical approach is clearly
demonstrated in synthesizing such items as effective mass,
impedance, spring rate, etc. from two basic expressions for
acceleration and force. A case in point is the impedance transfer
function which may be derived as follows:

IMPEDANCE $\qquad G_Z(S) = \dfrac{F_{input}}{V_{input}}(S)$

Since $\qquad A(S) = SV(S)$,

and Acceleration $\qquad G_A(S) = \dfrac{A_{head}}{A_{input}}(S)$ .

Force $\qquad G_F(S) = \dfrac{F_{input}}{A_{head}}(S)$ $\qquad$ (11) ,

$$G_Z(S) = SG_F(S)\, G_A(S)$$

Expanding the transfer function approach, the new criteria "absorbed power" was developed by Mr. R. A. Lee. It was shown that "absorbed power" corresponded to subjective response. Later, it was clearly indicated that subjective response can be forecasted using "absorbed power" and that the effect of time may be treated analytically.

Detailed examination of the "absorbed power" criteria indicated advantages not present in the "acceleration" measurement. Acceleration is a vector quantity. It must be assessed in combination with frequency spectrum, magnitude, and direction. "Absorbed power", however, is a scalar quantity which may be described by magnitude only. Thus it is additive and may be summed in multi-degree-of-freedom environments.

Acceleration data measurements are usually collected at input locations where the influence of individual differences, posture, or seat cushions cannot be separately assessed. In this important area where acceleration demonstrates insensitivity, "absorbed power" displays sensitivity and varies with experimental and anatomical differences.

Figure 6 shows "absorbed power" versus ride. The thirteen rides plotted on the abscissa are random records. Each ride represents different descriptions of energy within the same frequency spectrum. It can be seen that the power which was analytically determined corresponds closely to the experimentally measured "absorbed power". Figure 7 describes the application of Equation 10 and compares the reciprocal of "absorbed power" (increased magnitude decreased severity) to the time of exposure. Figure 7 reveals the utility of this analytical approach to define time of exposure as well as ride severity.

Thus, the following distinct and separate conclusions are drawn:

1. Human response to sinusoidal and random vibration displays linear characteristics under physical equilibrium.

2. Transfer function statements accurately describe human response to random vibration.

3. Human vibration may be studied analytically using transfer function statements.

4.  "Absorbed power" describes human response quantitatively and is sensitive to time.

5.  Experimental determination of "absorbed power" does not require frequency spectrum analysis.

6.  Optimization studies involving human dynamics may be conducted continuously without time lapse that occurs for frequency spectrum analysis.

ACKNOWLEDGEMENTS:

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## Appendix A

### TABLE I - Statistical Data

|  | Average | Standard Deviation |
|---|---|---|
| Age (years) | 28.00 | 10.36 |
| **Height (inches)** | | |
| Height, stature | 70.27 | 2.64 |
| 1A. Sitting height, erect | 35.70 | 1.92 |
| 1B. Eye height, normal sitting, canthus | 30.90 | 2.05 |
| 1C. Buttock-shoulder height | 23.81 | 1.76 |
| 1D. Buttock-elbow height | 9.76 | 1.62 |
| 1E. Seat-height | 16.69 | 0.98 |
| 1F. Knee height | 22.66 | 0.98 |
| **Weight - no equipment (pounds)** | 175.17 | 22.19 |
| 1. Sitting weight, erect | 148.78 | 11.19 |
| 2. Sitting weight, braced | 114.71 | 21.54 |
| **Trunk (inches)** | | |
| 2A. Shoulder width | 18.33 | 0.93 |
| 2B. Elbow width | 19.47 | 1.50 |
| 2C. Seat width | 14.86 | 1.28 |
| 3A. Chest depth | 9.76 | 1.18 |
| 3B. Abdominal depth | 9.18 | 0.81 |
| **Head (inches)** | | |
| Head length, front to back | 7.73 | 0.29 |
| Head width, side to side | 6.26 | 0.21 |
| **Hand (inches)** | | |
| 7A Hand length | 7.93 | 0.51 |

7B  Hand width                          3.69            0.31

Arm (inches)

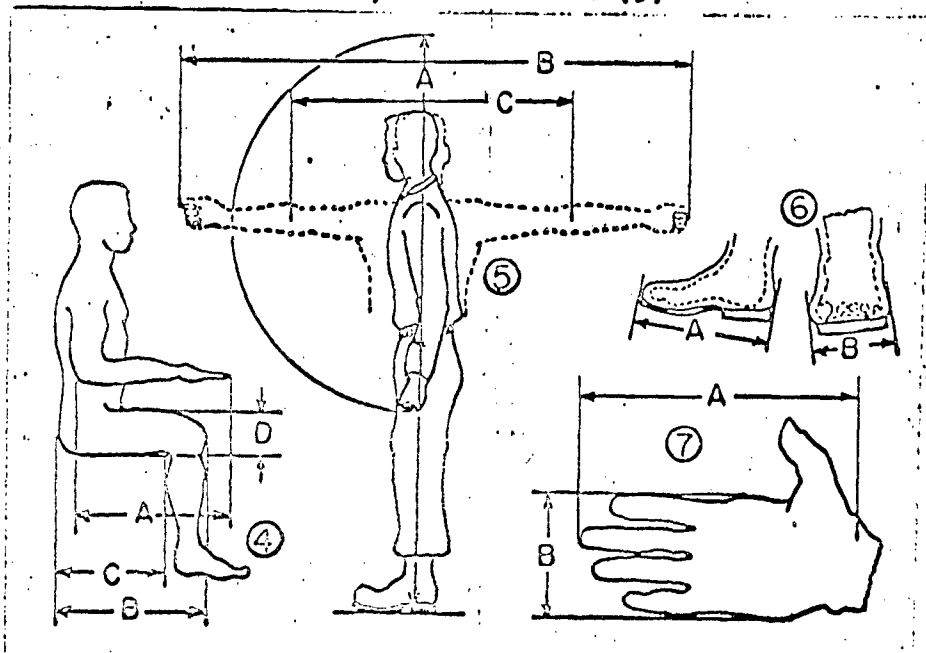4A  Elbow - finger length              19.34           0.92

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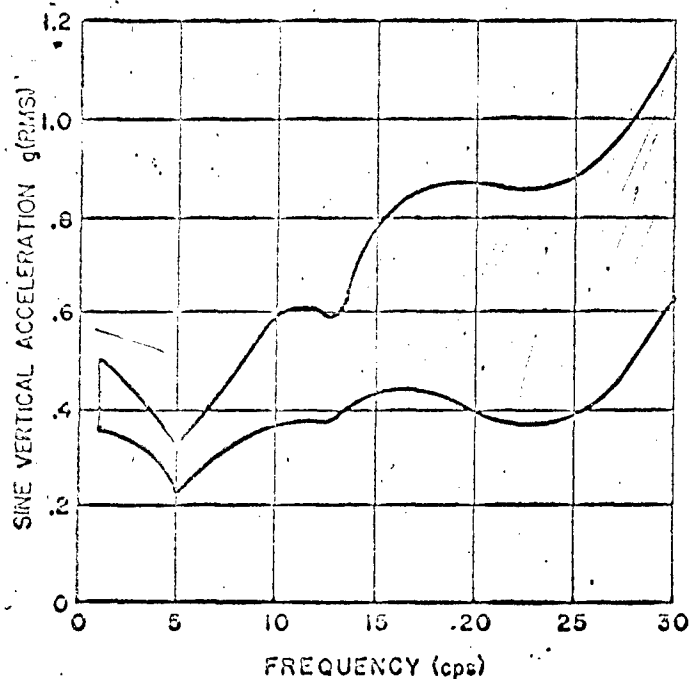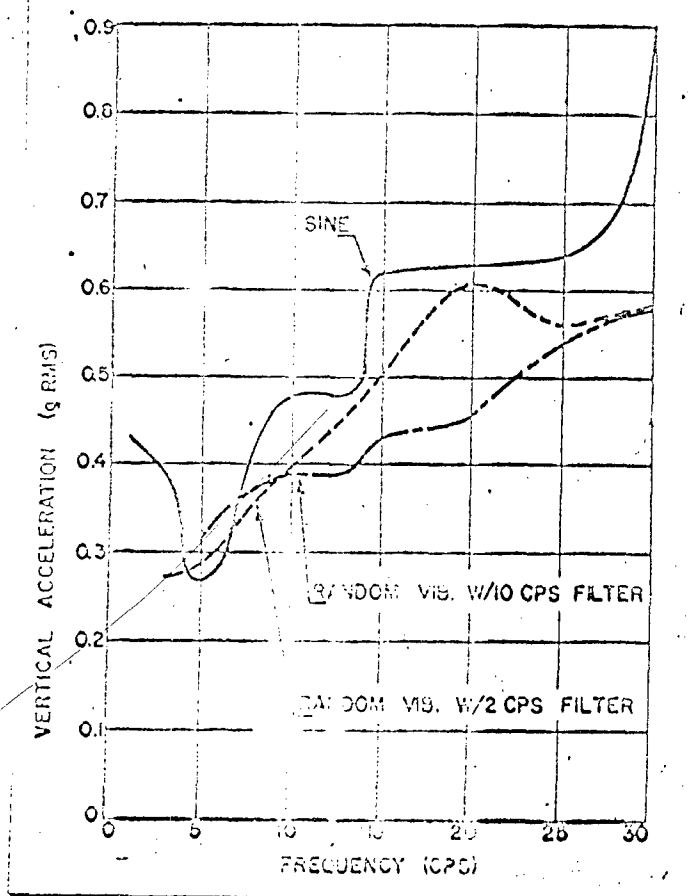## APPENDIX A: Statistical Data of Test Population
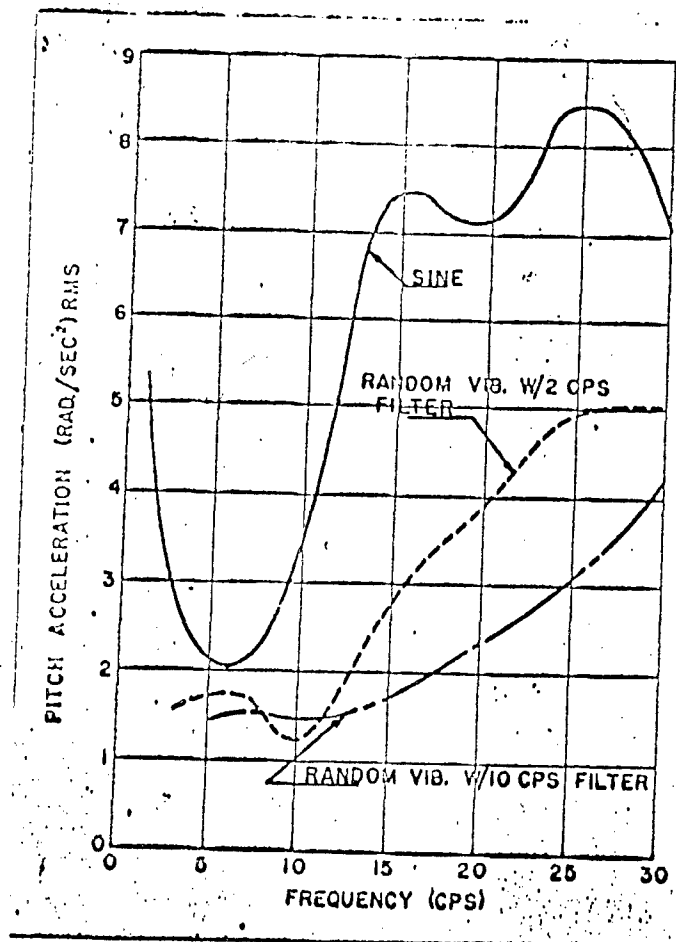


Body Measurements (3)



Body Measurements (3)

FIGURE 1



FIGURE 2



FIGURE 3

FIGURE 5

FIGURE 4

FIGURE 6

ABSORBED POWER COMPARISON

DIGITAL SOLUTION
ACTUAL VALUE

POWER (WATTS)

RIDE NUMBER



FIGURE 7

ABSORBED POWER-30/P$_{ave}$ (WATT$^{-1}$)
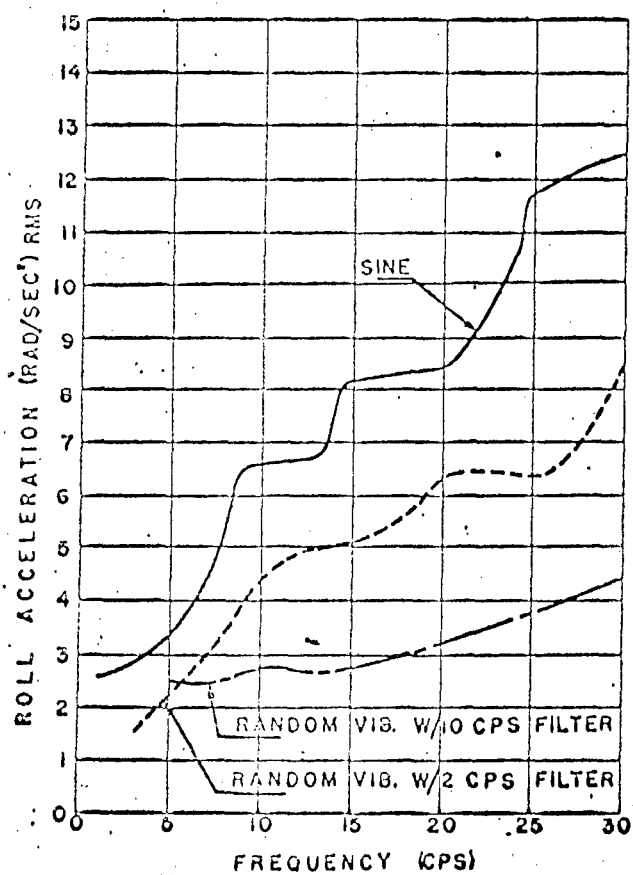
TIME (MIN)

RECIPROCAL OF
ABSORBED POWER

TIME OF
EXPOSURE

RIDE NUMBER
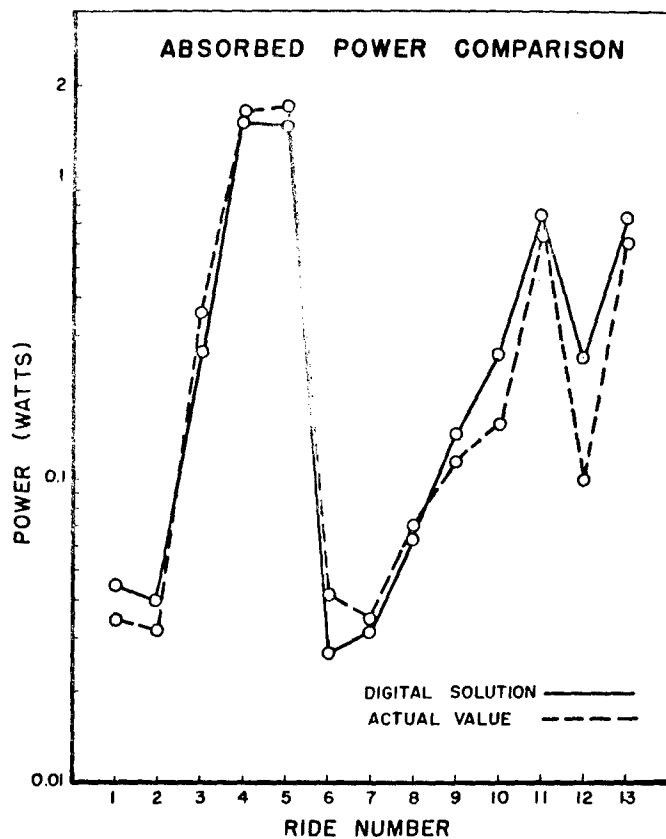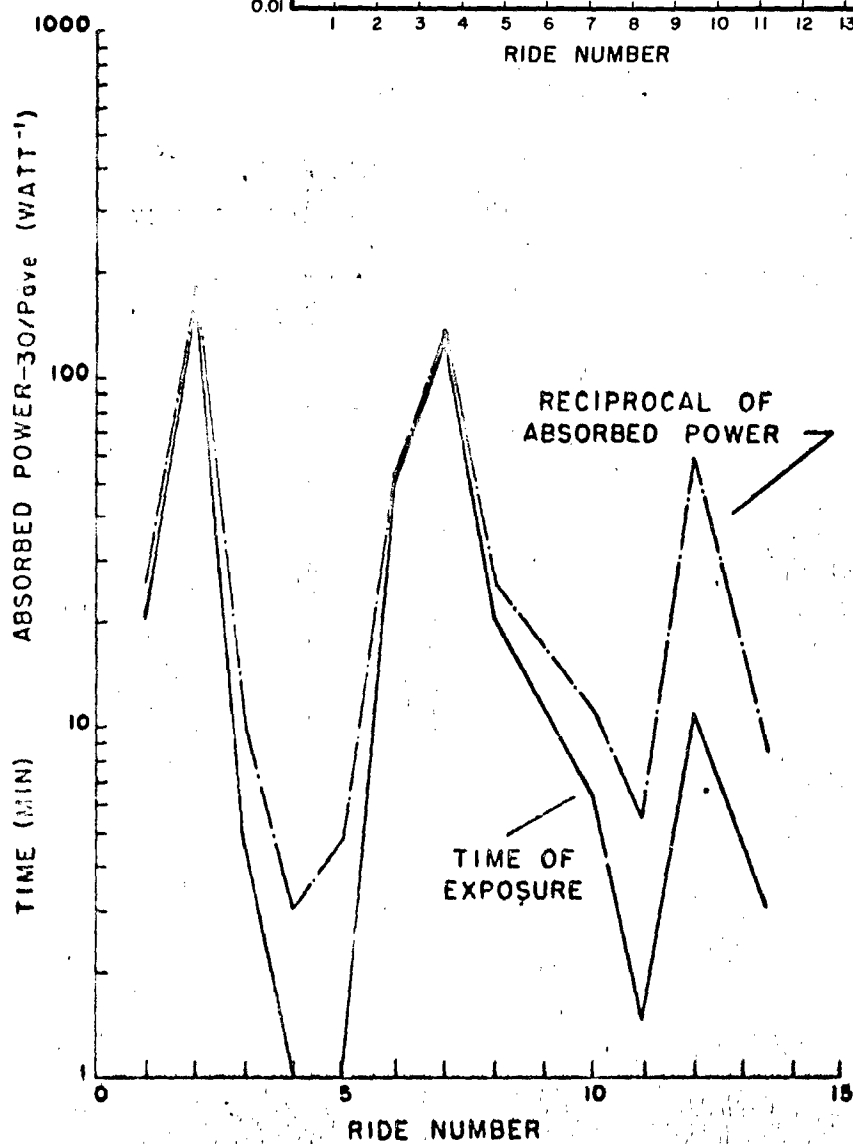
A-6

APPENDIX B  Derivation of Equations

The equation for RMS of acceleration is:

$$A_{rms} = \sqrt{\lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} a^2(t)\, dt} \qquad (1)$$

where $a(t)$ is any acceleration as a function of time and $A_{rms}$ is the RMS value of $a(t)$.

If displacement is sinusoidal and is given by

$$s = \frac{A_p}{\omega^2} \sin \omega t$$

Then velocity is given by

$$\frac{ds}{dt} = V(t) = \frac{A_p}{\omega} \cos \omega t \qquad (2)$$

where $A_p$ is the peak amplitude of the accelerations (from zero to peak) then the acceleration for this velocity is

$$\frac{dv}{dt} = a(t) = -A_p \sin \omega t, \qquad (3)$$

and the RMS of this sinusoidal acceleration from (1) and (3) is

$$A_{rms} = \sqrt{\lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} A_p^2 \sin^2 \omega t\, dt} \qquad (4)$$

Squaring both sides

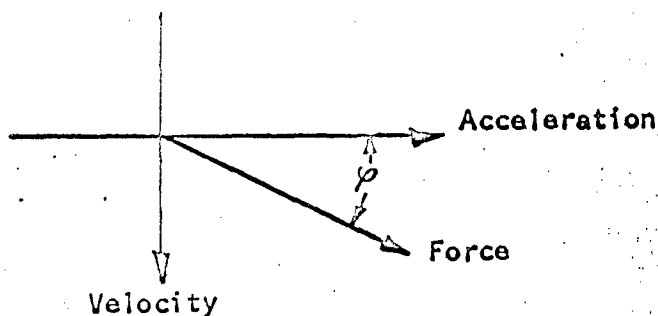$$A_{rms}^2 = \lim_{T \to \infty} \frac{A_p^2}{T} \int_{0}^{T} \sin^2 \omega t\, dt \qquad (5)$$

A generalized expression for average power (P) delivered by a system is "the time rate of doing work" and may be expressed as

$$P = \lim_{T \to \infty} \frac{1}{T} \int_0^T F(t)\; V(t)\; dt \qquad (6)$$

For a linear system, sinusoidal excited, the force is a function of time:

$$F(t) = K_G A_p \sin(\omega t + \varphi) \qquad (7)$$

where $\varphi$ is the angle between force and acceleration and $K_G$ is the gain factor.



Both $\varphi$ and $K_G$ are functions of $\omega$ but are constant at one frequency.

Substituting Equation (2) and Equation (7) into (6) power may be expressed as:

$$P = \lim_{T \to \infty} \frac{1}{T} \int_0^T \left[ K_G A_p \sin(\omega t + \varphi)\; \frac{A_p}{\omega} \cos \omega t \right] dt \qquad (8)$$

Let

$$\sin(\omega t + \varphi) = \sin \omega t \cos \varphi + \cos \omega t \sin \varphi \qquad (9)$$

Then:

$$P = \lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{K_G A_p^2}{\omega} (\sin\omega t \cos\omega t \cos\varphi + \cos^2\omega t \sin\varphi) \, dt \qquad (10)$$

Now $\int \sin^2\omega t \, dt = \frac{1}{2}t - \frac{1}{4\omega}\sin 2\omega t$ and $\int \cos^2\omega t \, dt = \frac{1}{2}t + \frac{1}{4\omega}\sin 2\omega t$

Therefore

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \sin^2\omega t \, dt = \lim_{T \to \infty} \frac{1}{T} \int_0^T \cos^2\omega t \, dt \qquad (11)$$

and since

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \sin\omega t \cos\omega t = \lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{1}{2}\sin 2\omega t \, dt = \qquad (12)$$

$$\lim_{T \to \infty} \frac{1}{T} [-\frac{1}{4\omega} \cos 2\omega t]_0^T = 0$$

the equation for average power becomes

$$P = \frac{K_G \sin\varphi}{\omega} \left[ \lim_{T \to \infty} \frac{A_p^2}{T} \int_0^T \sin^2\omega t \right] \qquad (13)$$

The term in the brackets of Equation (13) is the same as Equation (5) and is equal to $A^2_{rms}$.

Therefore, Equation (13) can be written

$$P = \frac{K_G \sin\varphi}{\omega} A^2_{rms} \qquad (14)$$

Since $K_G$ and $\varphi$ are constant at one frequency, Equation (14) can be written

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$$P_i = K_i A^2_{i_{rms}} \qquad (15)$$

where the subscript i indicates the value at a discrete frequency and

$$K_i = \frac{K_{G_i} \sin \varphi_i}{\omega_i} \qquad (16)$$

If $K_i$ is known for each frequency, "absorbed power" can be calculated when the RMS of a sinusoidal acceleration is known.

Vibration environments are generally defined by frequency content, either statistically or in the time domain. Field measurements are also generally reduced to this format. Thus, "absorbed power" should be determinable when the input data is in the form of a vibration spectrum, either RMS of acceleration or as power spectral density. "Absorbed Power", P, may be written as the integral:

$$P = \lim_{T \to \infty} \frac{1}{T} \int_0^T F(t)\, V(t)\, dt \qquad (17)$$

where force is given by

$$F(t) = \sum_{i=1}^{n} F_i \sin(\omega_i t + \varphi_i) \qquad (18)$$

and the velocity is

$$V(t) = \sum_{i=1}^{n} V_i \cos \omega_i t \qquad (19)$$

Using the identity

$$\sin(\omega t + \varphi) = \sin\omega t \cos\varphi + \cos\omega t \sin\varphi \tag{20}$$

Force can be written as

$$F(t) = \sum_{i=1}^{n} (F_i \cos\varphi_i) \sin\omega_i t + \sum_{i=1}^{n} (F_i \sin\varphi_i) \cos\omega_i t \tag{21}$$

Taking the product of force and velocity

$$F(t)V(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} F_i V_j \cos\varphi_i (\sin\omega_i t \cos\omega_j t)$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} F_i V_j \sin\varphi_i (\cos\omega_i t \cos\omega_j t) \tag{22}$$

and substituting Equation (22) into Equation (17) yields

$$P = \sum_{i=1}^{n} \sum_{j=1}^{n} F_i V_j \cos\varphi_i \left[ \lim_{T \to \infty} \frac{1}{T} \int_0^T \sin\omega_i t \cos\omega_j t \, dt \right]$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} F_i V_j \sin\varphi_i \left[ \lim_{T \to \infty} \frac{1}{T} \int_0^T \cos\omega_i t \cos\omega_j t \, dt \right] \tag{23}$$

Using the identities

$$\sin at \cos bt = \frac{1}{2} \left[ \sin(a-b)t + \sin(a+b)t \right]$$

$$\cos at \cos bt = \frac{1}{2} \left[ \cos(a-b)t + \cos(a+b)t \right]$$

one obtains

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \sin at \cos bt \, dt$$

$$= \lim_{T \to \infty} \frac{1}{T} \left[ -\frac{\cos(a-b)t}{2(a-b)} - \frac{\cos(a+b)t}{2(a+b)} \right]_0^T = 0 \qquad\qquad (24)$$

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \cos at \cos bt \, dt$$

$$= \lim_{T \to \infty} \frac{1}{T} \left[ \frac{\sin(a-b)t}{2(a-b)} + \frac{\sin(a+b)t}{2(a+b)} \right]_0^T = 0 \qquad a \neq b \qquad (25)$$

and since

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \cos \omega_i t \cos \omega_j t \, dt$$

$$= \lim_{T \to \infty} \frac{1}{T} \int_0^T \cos^2 \omega_j t \, dt; \quad i = j \qquad\qquad (26)$$

and Equation (12) holds, Equation (23) becomes

$$p = \sum_{i=1}^{n} F_i V_i \sin \varphi_i \left[ \lim_{T \to \infty} \frac{1}{T} \left( \frac{T}{2} + \frac{\sin 2\omega_i T}{4\omega_i} \right) \right]$$

$$p = \sum_{i=1}^{n} F_i V_i \sin \varphi_i \left[ \lim_{T \to \infty} \frac{1}{2} + \frac{1}{T} \left( \frac{\sin 2\omega_i T}{4\omega_i} \right) \right] \qquad\qquad (27)$$

Taking the limits one obtains

$$p = \frac{1}{2} \sum_{i=1}^{n} F_i V_i \sin \psi_i \tag{28}$$

Since for sinusoidal waves

$$F_i = K_{G_i} A_{p_i} \qquad V_i = A_{p_i}/\omega_i \text{ and } A_p^2 = 2A_{rms}^2 \text{ }^* \tag{29}$$

and using equation (16),

Equation(28) can be written

$$P = \sum_{i=1}^{n} K_i A_i^2{}_{rms} \tag{30}$$

From (5)

$$^* \quad A_{rms}^2 = \lim_{T \to \infty} \frac{1}{T} \int_0^T A_p^2 \sin^2 \omega t \, dt$$

$$A_{rms}^2 = \lim_{T \to \infty} \frac{1}{T} \left[ \frac{A_p^2 t}{2} - \frac{A_p^2}{4\omega} \sin 2\omega t \right]_0^T$$

$$A_{rms}^2 = \lim_{T \to \infty} \left[ \frac{A_p^2}{2} - \frac{A_p^2}{4\omega T} \sin 2\omega T \right]$$

$$A_{rms}^2 = \frac{A_p^2}{2} \text{ or } A_p^2 = 2A_{rms}^2$$

Therefore, the total "absorbed power" for a vibration spectrum is the summation of the power at each frequency. The power at each frequency can be computed by multiplying the acceleration squared by its appropriate constant ($K_i$).

Derivation of $K_i$:

If $G(S)$ is a polynomial of the form

$$G(S) = K_o \left[ \frac{S^n + C_1 S^{n-1} + \ldots + C_{n-1} \ S + C_n}{S^m + C_{n+1} \ S^{m-1} + \ldots + C_{n+m-1} \ S + C_{n+m}} \right] \quad (31)$$

letting $S \rightarrow J\omega$, one obtains

$$(32)$$

$$G(J\omega) = K_o \left[ \frac{(J\omega)^n + C_1 (J\omega)^{n-1} + \ldots + C_{n-1} (J\omega) + C_n}{(J\omega)^m + C_{n+1}(J\omega)^{m-1} + \ldots + C_{n+m-1} (J\omega) + C_{n+m}} \right].$$

The numerator and denominator of Equation (32) can be separated into real and imaginary parts.

If the exponent $r$ is even, the real and imaginary parts are given by

$$R_e = (-1)^{r/2} \left[ \omega^r - K_2 \omega^{r-2} + K_4 \omega^{r-4} - K_6 \omega^{r-6} + \ldots \right] \quad (33)$$

$$I_m = (-1)^{r/2} \omega \left[ -K_1 \omega^{r-2} + K_3 \omega^{r-4} - K_5 \omega^{r-6} + K_7 \omega^{r-8} + \ldots \right] \quad (34)$$

If the exponent is odd, the form is

$$R_e = (-1)^{\frac{r-1}{2}} \left[ K_1\omega^{r-1} - K_3\omega^{r-3} + K_5\omega^{r-5} - K_7\omega^{r-7} + \ldots \right] \quad (35)$$

$$I_m = (-1)^{\frac{r-1}{2}} \omega \left[ \omega^{r-1} - K_2\omega^{r-3} + K_4\omega^{r-5} - K_6\omega^{r-7} + \ldots \right] \quad (36)$$

Where r is the degree of the polynomial in either the numerator or denominator.

Writing Equation (32) in the form

$$G(J\omega) = \frac{K_o (F_1 + J\omega F_2)}{F_3 + J\omega F_4} \quad (37)$$

where the F's are given by Equations (33) thru (36)

Multiplying denominator and numerator of Equation (37) by $(F_3 - J\omega F_4)$ yields

$$G(J\omega) = K_o \left[ \frac{F_1 F_3 + \omega^2 F_2 F_4 + J\omega (F_2 F_3 - F_1 F_4)}{F_3^2 + \omega^2 F_4^2} \right] \quad (38)$$

G(Jω) may be thought of as being in the form

$$G(J\omega) = x + yi$$

where $x = K_o \dfrac{F_1 F_3 + \omega^2 F_2 F_4}{F_3^2 + \omega^2 F_4^2}$

$$y = K_o \frac{\omega (F_2 F_3 - F_1 F_4)}{F_3^2 + \omega^2 F_4^2}$$

The number $\gamma = \sqrt{x^2 + y^2}$ is called the modulus or absolute value of the complex number. The smallest positive angle which the vector makes with the positive x-axis is represented by Arctan $\frac{y}{x}$.

Hence

$$\gamma = \left| G(J\omega) \right| = \sqrt{K_o^2 \frac{(F_1F_3 + \omega^2 F_2F_4)^2}{(F_3^2 + \omega^2 F_4^2)^2} + K_o^2 \frac{\omega^2(F_2F_3 - F_1F_4)^2}{(F_3^2 + \omega^2 F_4^2)^2}}$$

$$\left| G(J\omega) \right| = K_o \sqrt{\frac{(F_1F_3 + \omega^2 F_2F_4)^2 + \omega^2(F_2F_3 - F_1F_4)^2}{(F_3^2 + \omega^2 F_4^2)^2}}$$

$$\left| G(J\omega) \right| = K_o \frac{\sqrt{(F_1F_3 + \omega^2 F_2F_4)^2 + \omega^2(F_2F_3 - F_1F_4)^2}}{(F_3^2 + \omega^2 F_4^2)}$$

(39)

The angle which the vector makes with the x-axis is

$$\mathcal{Y}_i = \text{Arctan}\left[ \frac{-\omega(F_2F_3 - F_1F_4)}{F_1F_3 + \omega^2 F_2F_4} \right]$$

Equation (39) may be simplified as follows

$$\left| G(J\omega) \right| = \frac{\sqrt{F_1^2F_3^2 + 2\omega^2 F_1F_2F_3F_4 + \omega^4 F_2^2F_4^2 + \omega^2 F_2^2F_3^2 - 2\omega^2 F_1F_2F_3F_4 + \omega^2 F_1^2F_4^2}}{F_3^2 + \omega^2 F_4^2}$$

$$|G(J\omega)| = \sqrt{\frac{F_1^2(F_3^2 + \omega^2 F_4^2) + \omega^2 F_2^2(\omega^2 F_4^2 + F_3^2)}{F_3^2 + \omega^2 F_4^2}}$$

$$|G(J\omega)| = \sqrt{\frac{(F_3^2 + \omega^2 F_4^2)(F_1^2 + \omega^2 F_2^2)}{F_3^2 + \omega^2 F_4^2}}$$

$$|G(J\omega)| = \sqrt{\frac{F_1^2 + \omega^2 F_2^2}{F_3^2 + \omega^2 F_4^2}} \qquad (40)$$

If $G(J\omega)$ is a transfer function defined by

$$\frac{Force}{Acceleration} = G(J\omega) \qquad (41)$$

then

$$F_{i_{rms}} = A_{i_{rms}} G(J\omega_i) \qquad (42)$$

Using Equation (28) and the relationship given in Equation (29), "absorbed power" can be written as

$$P = \sum_{i=1}^{n} \frac{1}{2}(F_i A_{p_i}/\omega_i) \sin \varphi_i \qquad (43)$$

Substituting Equation (42) into Equation (43),

$$P = \sum_{i=1}^{n} \left[ \frac{|G(J\omega_i)| A_{i_{rms}}^2}{\omega_i} \right] \sin \varphi_i \qquad (44)$$

This is the same as Equation (30) with

$$K_i = \frac{|G(J\omega_i)| \sin \varphi_i}{\omega_i} \qquad (45)$$

where

$$|G(J\omega)| = K_o \sqrt{\frac{F_1{}^2 + \omega^2 F_2{}^2}{F_3{}^2 + \omega^2 F_4{}^2}} \tag{46}$$

And

$$\sin\varphi = \frac{\omega(F_1 F_4 - F_2 F_3)}{\sqrt{(F_1 F_3 + \omega^2 F_2 F_4)^2 + \omega^2 (F_1 F_4 - F_2 F_3)^2}}$$

Squaring and cancelling yields

$$\sin\varphi = \frac{\omega(F_1 F_4 - F_2 F_3)}{\sqrt{F_1{}^2(F_3{}^2 + \omega^2 F_4{}^2) + \omega^2 F_2{}^2(\omega^2 F_4{}^2 + F_3{}^2)}}$$

$$\sin\varphi = \frac{\omega(F_1 F_4 - F_2 F_3)}{\sqrt{(\omega^2 F_4{}^2 + F_3{}^2)(F_1{}^2 + \omega^2 F_2{}^2)}} \tag{47}$$

Substituting the relationship for $|G(J\omega)|$ and $\sin\varphi$ into Equation (45) yields

$$K_i = K_o \left[ \frac{F_1 F_4 - F_2 F_3}{F_3{}^2 + \omega^2 F_4{}^2} \right] \tag{48}$$

Using Equation (48) and Equation (30), an "absorbed power" can be calculated for any given frequency spectrum.

$$P = \sum_{i=1}^{n} K_o \left[ \frac{F_1 F_4 - F_2 F_3}{F_3{}^2 + \omega^2 F_4{}^2} \right] A_{i_{rms}}{}^2 \tag{49}$$

Solving for $A_i$ in Equation (30) gives

$$A_i = \sqrt{\frac{P}{K_i}}$$ (50)

For any given power and frequency, an acceleration can be calculated.

For vibrations of short duration, "absorbed power" can be used as a measurement of vibration severity. To include fatigue, the energy absorbed is modified to include the time at which fatigue begins.

$$P_t = P + \frac{1}{t} \int_0^t P \; dt$$ (51)

where $P_t$ is the long-term "absorbed power", $P$ is the average power, $\frac{1}{t}$ is the time scale factor, the initial time $t_o$ is the onset of fatigue. Time $t$ is the length of exposure.

Power need not remain constant but can be a function of time. If power remains constant, we obtain

$$P_t = P + P \frac{t}{t_o}$$ (52)

Substituting $KA^2$ for power,

$$P_t = KA^2 + KA^2 \frac{t}{t_o}$$ (53)

And an acceleration-time relationship is given by

$$A = \sqrt{\frac{\frac{P_t}{K}}{1 + \frac{t}{t_o}}}$$ (54)